

EPHOU-05-002
 KEK-TH-1004
 Imperial/TP/050402
 UTHEP-502
 hep-th/0504123

Quantum fluctuations of rotating strings in $AdS_5 \times S^5$

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Abstract

We discuss quantum fluctuations of a class of rotating strings in $AdS_5 \times S^5$. In particular, we develop a systematic method to compute the one-loop sigma-model effective actions in closed forms as expansions for large spins. As examples, we explicitly evaluate the leading terms for the constant radii strings in the $SO(6)$ sector with two equal spins, the $SU(2)$ sector, and the $SL(2)$ sector. We also obtain the leading quantum corrections to the space-time energy for these sectors.

April 2005

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1 Introduction

To better understand dynamical aspects of the AdS/CFT correspondence, one may need studies beyond the BPS sectors. An important step toward this direction was made in [1]. Rotating strings in $AdS_5 \times S^5$ provide its generalizations, where deeply non-BPS sectors can be probed [2]-[9]. (For a review, see [10, 11].) In particular, for a certain class of rotating strings, one can find an exact agreement between unprotected quantities on the string and the gauge theory sides, which is not necessarily guaranteed by the AdS/CFT correspondence. Moreover, the correspondence has been extended, in a unified manner, to that of effective theories or general solutions [12]-[15].

A clue for understanding this agreement may be the integrable structures on both sides (see, e.g., [3, 8], [14]-[20]). A role of a certain asymptotic (“nearly”) BPS condition has also been pointed out [21]. For large spins, the world-sheets of the rotating strings form nearly null surfaces and the strings become effectively tensionless [21, 22], which intuitively means that the constituents of the strings appear to be free [23]. However, we do not yet know why we find the exact agreement, and why it starts to break down at a certain level [24].

In this development, the analysis on the string side tends to be classical because of difficulties in the quantization, despite that information of the quantized strings is necessary to complete the correspondence. This contrasts with detailed quantum analysis on the gauge theory side [11]. As for the quantum aspects of the rotating strings,¹ the one-loop sigma-model fluctuations and their stability have been studied for the “constant radii” strings [4, 20, 29]. Based on these results, the one-loop corrections to the space-time energy have been studied numerically for the $SO(6)$ and the $SU(2)$ sectors with two equal spins [30], and for the $SL(2)$ sector [31]. Furthermore, the leading correction for the $SL(2)$ sector has been matched in a closed form with the finite size correction of the anomalous dimension on the gauge theory side [32]. This result can also be extrapolated to the $SU(2)$ sector (with two equal spins), up to subtleties of instability.

In this paper, we discuss quantum fluctuations of the rotating strings in $AdS_5 \times S^5$. In particular, we develop a systematic method to compute the one-loop sigma-model effective actions and the corrections to the space-time energy in such backgrounds, so that they are obtained in closed forms as expansions for large spins. As examples, we consider the constant radii strings in the $SO(6)$ sector with two equal spins, the $SU(2)$ sector, and the $SL(2)$ sector, and explicitly evaluate the leading terms in the expansion. We note that it is in principle possible to carry out the expansion up to any given order. The asymptotic BPS condition and the effective tensionless limit seem to be characteristic of the correspondence and useful for understanding the string in $AdS_5 \times S^5$ itself. However, their consequences for the quantum string have not been investigated well. Through concrete computations, we can see how they work to make the quantum corrections subleading for large spins.

¹For related works, see, e.g., [25]-[28].

The organization of this paper is as follows. In section 2, we summarize necessary ingredients to discuss the one-loop fluctuations of strings in $AdS_5 \times S^5$. In section 3, we discuss the quadratic fluctuations of the constant radii strings in the $SO(6)$ sector with two equal spins. We evaluate the fluctuation operators in rotated functional bases, so that the expansion for large spins becomes well-defined. In the course, we obtain the fermionic fluctuation operator for the generic $SO(6)$ sector. In section 4, we develop a large J (total spin) expansion of the one-loop effective action. We explicitly evaluate the leading term for large J (up to and including $\mathcal{O}(1/J)$) in a closed form. Essentially the same procedures are applied to other sectors in the following sections. We briefly summarize the results for the $SU(2)$ sector in section 5, and for the $SL(2)$ sector in section 6. At the leading order, all the one-loop effective actions take a universal form, which is proportional to a geometric invariant. In section 7, from the one-loop effective actions, we read off the corrections to the space-time energy up to and including $\mathcal{O}(1/J^2)$. Comparing the results with the finite size corrections to the anomalous dimension on the gauge theory side [32]-[36], we find that the dependence on the winding numbers and the filling fractions agree with that of the “non-anomalous” (zero-mode) part on the gauge theory side. Relation to the earlier results in the literature is also discussed. We conclude in section 8. In the appendix, we summarize how to evaluate a constant which appears in the expression of the one-loop effective actions.

2 Preliminaries

We consider one-loop sigma-model fluctuations of a certain class of rotating strings in type IIB theory. Here, we summarize our notation and ingredients used in the following sections.

Coordinates

The metric of $AdS_5 \times S^5$ takes a form

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\nu = ds_{AdS_5}^2 + ds_{S^5}^2, \\ -ds_{AdS_5}^2 &= d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho (d\theta^2 + \cos^2 \theta d\phi_4^2 + \sin^2 \theta d\phi_5^2), \\ -ds_{S^5}^2 &= d\gamma^2 + \cos^2 \gamma d\phi_3^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\phi_1^2 + \sin^2 \psi d\phi_2^2). \end{aligned} \quad (2.1)$$

To express the rotating string solutions, it is useful to introduce complex variables Z_r ($r = 0, \dots, 5$), so that AdS_5 and S^5 are expressed as hypersurfaces in flat spaces, $|Z_0|^2 - |Z_4|^2 - |Z_5|^2 = 1$ and $|Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1$, respectively. In terms of the above coordinates, one can set

$$Z_r = a_r e^{i\phi_r}, \quad (2.2)$$

with $\phi_0 = t$ and

$$\begin{aligned} a_1 &= \sin \gamma \cos \psi, & a_2 &= \sin \gamma \sin \psi, & a_3 &= \cos \gamma, \\ a_4 &= \sinh \rho \cos \theta, & a_5 &= \sinh \rho \sin \theta, & a_0 &= \cosh \rho. \end{aligned} \quad (2.3)$$

Bosonic fluctuation

In the conformal gauge, the bosonic part of the world-sheet action is given by

$$S_B = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \eta^{ij} G_{\mu\nu} \partial_i x^\mu \partial_j x^\nu, \quad (2.4)$$

where $i, j = (\tau, \sigma)$, and $\eta_{\tau\tau} = -\eta_{\sigma\sigma} = +1$. A simple way to obtain the fluctuation Lagrangian is the geodesic expansion [37]. Introducing a vector y^a in the tangent space of the space-time, the quadratic fluctuation Lagrangian is given by

$$L_B^{(2)} = -\frac{1}{2} \eta_{ab} D_i y^a D^i y^b - \frac{1}{2} y^a y^b e_i^c e^{i d} R_{acbd}. \quad (2.5)$$

η_{ab} is the flat ten-dimensional metric with mostly minus signatures. D_i and e_i^a are the projections of the covariant derivative D_μ and the vielbein e_μ^a , respectively; for example, $(D_i y)^a = \partial_i x^\mu (\partial_\mu + \omega_{\mu,b}^a) y^b$ with $\omega_{\mu,ba}$ the connection one-form. For $AdS_5 \times S^5$, the curvature R_{abcd} is simple:

$$R_{abcd} = \mp (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc}), \quad (2.6)$$

with the minus sign if all indices (a, b, c, d) correspond to AdS_5 and the plus sign if (a, b, c, d) to S^5 ; otherwise it vanishes. In term of the global coordinate system in (2.1), the non-vanishing $\omega_{i,ab} (= \partial_i x^\mu \omega_{\mu,ab})$ are, up to the anti-symmetry,

$$\begin{aligned} \omega_{i,\theta\rho} &= \cosh \rho \partial_i \theta, & \omega_{i,t\rho} &= -\sinh \rho \partial_i t, & \omega_{i,\phi_4\rho} &= \cosh \rho \cos \theta \partial_i \phi_4, \\ \omega_{i,\phi_4\theta} &= -\sin \theta \partial_i \phi_4, & \omega_{i,\phi_5\rho} &= \cosh \rho \sin \theta \partial_i \phi_5, & \omega_{i,\phi_5\theta} &= \cos \theta \partial_i \phi_5, \\ \omega_{i,\psi\gamma} &= \cos \gamma \partial_i \psi, & \omega_{i,\phi_3\gamma} &= -\sin \gamma \partial_i \phi_3, & \omega_{i,\phi_1\gamma} &= \cos \gamma \cos \psi \partial_i \phi_1, \\ \omega_{i,\phi_1\psi} &= -\sin \psi \partial_i \phi_1, & \omega_{i,\phi_2\gamma} &= \cos \gamma \sin \psi \partial_i \phi_2, & \omega_{i,\phi_2\psi} &= \cos \psi \partial_i \phi_2. \end{aligned} \quad (2.7)$$

Fermionic fluctuation

To study the fermionic fluctuations, we use the quadratic fermionic part of the type IIB Green-Schwarz (GS) action on $AdS_5 \times S^5$ [38]. Following the notation in [39], it is expressed as

$$\begin{aligned} L_F^{(2)} &= i\theta^I P_{IJ}^{ij} \mathcal{D}_{ijJK} \theta^K, \\ \mathcal{D}_{ijJK} &= \hat{\sigma}_i \left[(\partial_j - \frac{1}{4} \omega_{j,ab} \sigma^{ab}) \delta_{JK} - \frac{1}{4 \cdot 480} F_{abcde} \sigma^{abcde} \hat{\sigma}_j(\rho_0)_{JK} \right], \\ P_{IJ}^{ij} &= \eta^{ij} \delta_{IJ} + \epsilon^{ij} (\rho_3)_{IJ}, \quad \hat{\sigma}_j = e_j^a \bar{\sigma}_a, \quad \epsilon^{01} = +1, \\ \rho_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (2.8)$$

Here, $(\theta^I)^\alpha$ ($I = 1, 2$) are ten-dimensional Majorana-Weyl spinors with 16 components; $(\bar{\sigma}^a)_{\alpha\beta}, (\sigma^a)^{\alpha\beta}$ are 16×16 gamma matrices in ten dimensions; their anti-symmetrization is given, e.g., by $\sigma^{ab} = (\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a)/2$, $\bar{\sigma}^{ab} = (\bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a)/2$. If we label the AdS_5 and S^5 parts by $a, b = (0, 6, 7, 8, 9)$ and $(1, 2, 3, 4, 5)$, respectively, the non-vanishing components of the five-form are $F_{06789} = F_{12345} = 4$.

The GS action has the κ -symmetry, the relevant part of which is now

$$\delta_\kappa \vartheta^\alpha = (\hat{\sigma}_i)^{\alpha\beta} \eta^{ij} \kappa_{j\beta}, \quad (2.9)$$

where $\eta^{ij} \kappa_{j\beta} = \epsilon^{ij} \bar{\kappa}_{j\beta}$, $\vartheta = \theta^1 + i\theta^2$ and $\kappa_{j\beta} = \kappa_{j\beta}^1 + i\kappa_{j\beta}^2$. Following [29], we fix this symmetry by setting

$$\theta^1 = \theta^2. \quad (2.10)$$

One can check that this gauge is actually possible for the backgrounds which we consider in the following sections. In this gauge, the form of the quadratic Lagrangian is simplified to

$$\begin{aligned} L_F^{(2)} &= 2i\theta^1 D_F \theta^1, \\ D_F &= \eta^{ij} \hat{\sigma}_i (\partial_j - \frac{1}{4} \omega_{j,ab} \sigma^{ab}) - \frac{1}{2} \epsilon^{ij} \hat{\sigma}_i \sigma_* \hat{\sigma}_j, \end{aligned} \quad (2.11)$$

where $\sigma_* \equiv \sigma^{06789} = \sigma^{12345}$.

3 Three-spin rotating string in S^5

In this paper, we consider classes of the “constant radii” solutions [10], in which a_r in (2.2) are constant. Bearing in mind possible extensions to more general cases, we develop a method to compute one-loop effective actions in a large-spin expansion. In this and the next sections, we discuss it in some detail for the constant radii strings with three spins in S^5 , i.e., in the $SO(6)$ sector. We apply this method to other cases later.

3.1 Solution

The solution which we consider is

$$Z_0 = e^{i\kappa\tau}, \quad Z_s = a_s e^{i(w_s\tau + m_s\sigma)}, \quad (3.1)$$

where $s = 1, 2, 3$; κ, a_s, w_s are constant with $\sum_s a_s^2 = 1$; m_s are integers (when the period of σ is 2π). Other fields including fermionic ones vanish. The equations of motion and the Virasoro constraints give

$$\begin{aligned} w_s^2 &= \nu^2 + m_s^2 \quad (\text{if } a_s \neq 0), \\ \kappa^2 &= \sum a_s^2 (w_s^2 + m_s^2), \quad 0 = \sum a_s^2 w_s m_s, \end{aligned} \quad (3.2)$$

with ν a constant. Classically, the space-time energy and the three spins in S^5 are given by

$$E = \sqrt{\lambda}\kappa, \quad J_s = \sqrt{\lambda}a_s^2 w_s, \quad (3.3)$$

where $\sqrt{\lambda} = R^2/\alpha'$ and R is the radius of $AdS_5 \times S^5$.

When $J_1 = J_2$ and $m_3 = 0$, the parameters of the solution become

$$\begin{aligned} w_1 = w_2 &\equiv w, & m_1 = -m_2 &\equiv m, & a_1 = a_2 &= s_\gamma/\sqrt{2}, \\ w_3 &= \nu, & m_3 &= 0, & a_3 &= c_\gamma, \\ w^2 &= \nu^2 + m^2, & \kappa^2 &= \nu^2 + 2m^2 s_\gamma^2, \end{aligned} \quad (3.4)$$

where $s_x \equiv \sin x, c_x \equiv \cos x$. In the following, we call this simplified solution the $J_1 = J_2$ three-spin solution. The point-like (BMN) solution in [1] is also obtained by setting $a_{1,2} = w_{1,2} = m_{1,2} = m_3 = 0$ in (3.1) and (3.2).

3.2 Bosonic fluctuation

Now, let us consider the bosonic fluctuations around the three-spin solution in (3.1). Substituting the solution into (2.5), one finds that the fluctuations in the AdS_5 and the S^5 parts decouple. The contribution to the one-loop effective action from the AdS_5 part is then represented by the determinant of the quadratic operator,

$$D_{pq}^B = -\eta_{pq}\partial^2 + \kappa^2 R_{ptqt}, \quad (3.5)$$

where $p, q = (t, \rho, \theta, \phi_4, \phi_5)$ or $(0, 6, 7, 8, 9)$, and $\partial^2 = \eta^{ij}\partial_i\partial_j$. This operator describes one time-like massless boson and four space-like bosons with mass squared κ^2 , as in the BMN case. Taking the determinant with respect to the tangent space indices gives

$$\det D_{pq}^B = -\partial^2(\partial^2 + \kappa^2)^4. \quad (3.6)$$

For the S^5 part, the quadratic term of the connection one-form and the curvature term cancel each other, to give

$$\begin{aligned} D_{mn}^B &= \delta_{mn}\partial^2 + 2(\omega_{\tau,mn}\partial_\tau - \omega_{\sigma,mn}\partial_\sigma), \\ \det D_{mn}^B &= \partial^2 \left[(\partial^2)^4 + \left(\sum_{s=1}^3 (1 - a_s^2)\Omega_s^2 \right) (\partial^2)^2 + (a_1^2\Omega_2^2\Omega_3^2 + a_2^2\Omega_3^2\Omega_1^2 + a_3^2\Omega_1^2\Omega_2^2) \right], \end{aligned} \quad (3.7)$$

where $m, n = (\phi_1, \phi_2, \phi_3, \psi, \gamma)$ or $(1, 2, 3, 4, 5)$, and

$$\Omega_s = 2(w_s\partial_\tau - m_s\partial_\sigma). \quad (3.8)$$

This determinant has been obtained in [20]. Note that we have assumed here that none of a_s vanishes in order to use the constraints $w_s^2 = \nu^2 + m_s^2$.

Change of functional bases

Later, we explicitly evaluate the functional determinant in a large $J(= \sum J_s)$ expansion. It turns out that $\det D_{mn}^B$ in (3.7) has an inappropriate infrared behavior for this purpose. Here, we make an $SO(5)$ rotation to avoid it, concentrating on the $J_1 = J_2$ three-spin solution with (3.4).

First, we take an orthogonal matrix,

$$Q_n^m = \frac{1}{\sqrt{2v}} \begin{pmatrix} s_\gamma \nu & v & 0 & c_\gamma w & 0 \\ s_\gamma \nu & -v & 0 & c_\gamma w & 0 \\ \sqrt{2} c_\gamma w & 0 & 0 & -\sqrt{2} s_\gamma \nu & 0 \\ 0 & 0 & \sqrt{2} v & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} v \end{pmatrix}, \quad (3.9)$$

with $v^2 = w^2 - M^2$ and $M^2 = s_\gamma^2 m^2$. By a change of bases using Q_n^m , the connection one-form is brought into a standard form. Namely, defining $\hat{\omega}_{i,mn} \equiv Q_n^k \omega_{i,kl} Q_n^l$, we find that

$$\hat{\omega}_{\tau,mn} = \begin{pmatrix} 0 & & \\ & p(-w) & \\ & & p(-v) \end{pmatrix}, \quad p(x) = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}. \quad (3.10)$$

We further introduce

$$R_n^m(\tau) = \begin{pmatrix} 1 & & \\ & P(\tau w) & \\ & & P(\tau v) \end{pmatrix}, \quad P(x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}, \quad (3.11)$$

so that R_n^m satisfies

$$0 = \partial_\tau R_{mn} - \hat{\omega}_{\tau,mk} R_n^k. \quad (3.12)$$

Then, by an transformation of the form,

$$\hat{\mathcal{O}} \equiv (QR)^{-1} \mathcal{O} (QR), \quad (3.13)$$

the quadratic operator becomes

$$\begin{aligned} \hat{D}_{mn}^B &= R_m^{k'} Q_{k'}^k D_{kl}^B Q^l{}_{l'} R_n^{l'} \\ &= \delta_{mn} \partial^2 + M_{mn} - 2\rho_{\sigma,mn} \partial_\sigma, \end{aligned} \quad (3.14)$$

where $M_{mn} = \text{diag}(0, w^2, w^2, v^2, v^2)$ and $\rho_{\sigma,mn} = R_m^k \hat{\omega}_{\sigma,kl} R_n^l$. After some algebra, we also find that

$$\begin{aligned} \det \hat{D}_{mn}^B &= (\partial^2 + w^2)^2 (\partial^2 + v^2)^2 \partial^2 \\ &\quad + 4c_\gamma^2 k^2 \partial_\sigma^2 \partial^2 \left[\left(1 + \frac{w^2}{v^2}\right) (\partial^2 + w^2) (\partial^2 + v^2) + 4c_\gamma^2 k^2 \left(\frac{w}{v}\right)^2 \partial_\sigma^2 \right] \\ &\quad + 4s_\gamma^2 k^2 \left(\frac{\nu}{v}\right)^2 \partial_\sigma^2 (\partial^2 + v^2) \left[(\partial^2 + w^2) (\partial^2 + v^2) + 4c_\gamma^2 k^2 \partial_\sigma^2 \right]. \end{aligned} \quad (3.15)$$

3.3 Fermionic fluctuation

Now, let us move onto the fermionic part. In order to evaluate the one-loop determinant, it is useful to make a rotation of D_F , so that the kinetic term is simplified to take the form for two-dimensional fermions [40]. For this purpose, we introduce an element of $SO(1, 9)$,

$$Q^a_b = \begin{pmatrix} q^a_b & 0 \\ 0 & 1_{6 \times 6} \end{pmatrix}, \quad q^a_b = \frac{1}{M} \begin{pmatrix} \kappa & -W_s \\ 0 & M_s \\ -W & \kappa W_s/W \\ 0 & l_s M \end{pmatrix}, \quad (3.16)$$

where $a, b = (t, \phi_1, \phi_2, \phi_3, \dots)$; $W_s \equiv a_s w_s$, $M_s = a_s m_s$, $W^2 \equiv \sum_s W_s^2$, $M^2 \equiv \sum_s M_s^2$; $1_{6 \times 6}$ is the 6×6 unit matrix; $l_{s_1} = \epsilon_{s_1 s_2 s_3} M_{s_2} W_{s_3} / MW$. This was chosen so that $Q_{\tilde{i}}^a = e^{-\varphi} e_{\tilde{i}}^a$ for $\tilde{i} = (0, 1)$, where φ is defined by the proportionality coefficient between the induced metric, $h_{ij} = e_i^a e_j^b \eta_{ab}$, and the world-sheet metric through $h_{ij} = e^{2\varphi} \eta_{ij}$. In our case, $e^{2\varphi} = M^2$. We then transform D_F by the element of $SO(1, 9)$ corresponding to Q^a_b , which we denote by $S(Q)$. Since $S^{-1} \sigma_a S = \sigma_b Q^b_a$ and hence

$$S^{-1} \hat{\sigma}_i S = e^\varphi \delta_i^{\tilde{i}} \bar{\sigma}_{\tilde{i}}, \quad (3.17)$$

this transformation replaces $\hat{\sigma}_i$ with $\bar{\sigma}_i$ in the quadratic operator, to give a desired form. In the following, we do not distinguish the indices \tilde{i} and i . After some algebra, we then find that

$$S^{-1}(Q) D_F S(Q) = e^\varphi \hat{D}_F, \quad (3.18)$$

where

$$\begin{aligned} \hat{D}_F &= \eta^{ij} \bar{\sigma}_i \left[\partial_j - \frac{1}{4} \omega_{j,ab} (\sigma^{ab})' \right] - \frac{1}{2} \epsilon^{ij} e^\varphi \bar{\sigma}_i \sigma'_* \bar{\sigma}_j, \\ &= \bar{\sigma}^i \partial_i + W \bar{\sigma}^{345} + \frac{1}{2MW} \sum_{a=0,1} \sum_{b=2,3} \sum_{c=4,5} \alpha_{abc} \bar{\sigma}^{abc}, \end{aligned} \quad (3.19)$$

$(\sigma^{ab})' = S^{-1} \sigma^{ab} S$, $\sigma'_* = S^{-1} \sigma_* S$, and

$$\begin{aligned} \alpha_{024} &= -s_\gamma c_\gamma \kappa (c_\psi^2 m_1^2 + s_\psi^2 m_2^2 - m_3^2), & \alpha_{124} &= (c_\gamma / s_\gamma) \kappa m_3 w_3, \\ \alpha_{034} &= s_\gamma s_\psi c_\psi w_3 (m_1 w_2 - m_2 w_1), & \alpha_{134} &= s_\gamma s_\psi c_\psi m_3 (m_1 w_2 - m_2 w_1), \\ \alpha_{025} &= s_\gamma s_\psi c_\psi \kappa (m_1^2 - m_2^2), & \alpha_{125} &= s_\gamma s_\psi c_\psi \kappa (m_1 w_1 - m_2 w_2), \\ \alpha_{035} &= -s_\gamma c_\gamma [m_3 w_1 w_2 - w_3 (c_\psi^2 m_1 w_2 + s_\psi^2 m_2 w_1)], \\ \alpha_{135} &= s_\gamma c_\gamma [m_1 m_2 w_3 - m_3 (s_\psi^2 m_1 w_2 + c_\psi^2 m_2 w_1)]. \end{aligned} \quad (3.20)$$

In the above, we have assumed that none of a_s vanishes.

Since e^φ is constant, we have now only to evaluate the determinant (pfaffian) of \hat{D}_F . To proceed, we adopt the following explicit realization of the gamma matrices (just to evaluate the determinants with respect to the spinor indices):

$$\begin{aligned}\sigma^1 &= \tau_3 \otimes 1 \otimes 1 \otimes 1, & \sigma^4 &= \tau_2 \otimes \tau_1 \otimes 1 \otimes 1, & \sigma^7 &= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_3, \\ \sigma^2 &= \tau_1 \otimes 1 \otimes 1 \otimes 1, & \sigma^5 &= \tau_2 \otimes \tau_2 \otimes \tau_3 \otimes 1, & \sigma^8 &= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_1, \\ \sigma^3 &= \tau_2 \otimes \tau_3 \otimes 1 \otimes 1, & \sigma^6 &= \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes 1, & \sigma^9 &= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2,\end{aligned}\tag{3.21}$$

$\sigma^0 = \bar{\sigma}^0 = 1$, and $\bar{\sigma}^a = -\sigma^a$ ($a = 1, \dots, 9$), where τ_a are the Pauli matrices. With these gamma matrices, \hat{D}_F takes the form

$$\hat{D}_F = \begin{pmatrix} \Delta_F^+ & 0 \\ 0 & \Delta_F^- \end{pmatrix} \otimes 1.\tag{3.22}$$

Denoting by the same symbols the matrices which are obtained by extracting the first two matrices in the tensor products in (3.21), for example, $\sigma^1 \rightarrow \tau_3 \otimes 1$, one finds that

$$\Delta_F^\pm = \bar{\sigma}^i \partial_i \mp W \bar{\sigma}^{012} + \beta_{1\pm} \bar{\sigma}^{024} + \beta_{2\pm} \bar{\sigma}^{124} + \beta_{3\pm} \bar{\sigma}^{034} + \beta_{4\pm} \bar{\sigma}^{134},\tag{3.23}$$

where

$$\begin{aligned}\beta_{1\pm} &= \frac{1}{2MW}(\alpha_{024} \pm \alpha_{135}), & \beta_{2\pm} &= \frac{1}{2MW}(\alpha_{124} \pm \alpha_{035}), \\ \beta_{3\pm} &= \frac{1}{2MW}(\alpha_{034} \mp \alpha_{125}), & \beta_{4\pm} &= \frac{1}{2MW}(\alpha_{134} \mp \alpha_{025}).\end{aligned}\tag{3.24}$$

From this, it follows that

$$\begin{aligned}\det \Delta_F^\pm &= (\partial^2)^2 + 2W^2 \partial^2 + 2\left(\sum_{n=1}^4 \beta_{n\pm}^2\right)(\partial_\tau^2 + \partial_\sigma^2) + 4(\beta_{1\pm}\beta_{2\pm} + \beta_{3\pm}\beta_{4\pm})\partial_\tau\partial_\sigma \\ &+ \left(\sum_{n=1}^4 \beta_{n\pm}^2\right)^2 - 4(\beta_{1\pm}\beta_{2\pm} + \beta_{3\pm}\beta_{4\pm})^2 + 2(\beta_{1\pm}^2 - \beta_{2\pm}^2 - \beta_{3\pm}^2 + \beta_{4\pm}^2)W^2 + W^4.\end{aligned}\tag{3.25}$$

The final one-loop contribution of the fermionic sector is then represented by

$$\text{pf } \hat{D}_F = \det \Delta_F^+ \det \Delta_F^-.\tag{3.26}$$

Given this result, it would be interesting to generalize the analysis in [30, 32] to the generic three-spin constant radii solution.

Change of functional bases

As in the case of D_{mn}^B , it turns out that the form of $\det \Delta_F^-$ in (3.25) is not convenient for our purpose. Thus, we make a change of bases again. To this end, we introduce a 4×4 matrix

$$R(\alpha) = 1 \otimes P(\tau\alpha),\tag{3.27}$$

with $P(x)$ given in (3.11). One then finds that $\det R^{-1}(\alpha)\Delta_F^\pm R(\alpha)$ is given just by replacing $\beta_{3\pm}$ in (3.25) with $\beta_{3\pm} - \alpha$. With this in mind, we define

$$\hat{\Delta}_F^\pm \equiv R^{-1}(\beta_{3\pm})\Delta_F^\pm R(\beta_{3\pm}), \quad (3.28)$$

so that $\beta_{3\pm}$ in (3.25) are set to be zero. This transformation for Δ_F^\pm is not inevitable for our purpose, but simplifies later computations.

Now, we focus on the $J_1 = J_2$ three-spin solution. In this case,

$$\beta_{1\pm} = -\frac{c_\gamma m(\kappa \pm \nu)}{2W}, \quad \beta_{3\pm} = \frac{w(\nu \mp \kappa)}{2W}, \quad \beta_{2\pm} = \beta_{4\pm} = 0, \quad (3.29)$$

and hence

$$\det \hat{\Delta}_F^\pm = (\partial^2)^2 + 2W^2\partial^2 + 2\beta_{1\pm}^2(\partial_\tau^2 + \partial_\sigma^2) + (\beta_{1\pm}^2 + W^2)^2. \quad (3.30)$$

One can confirm that the original determinants in (3.25) with (3.29) reproduce the characteristic frequencies obtained in [29].²

4 One-loop effective action and large J expansion

In this section, based on the results in section 3, we consider the one-loop effective action of the GS string in the $J_1 = J_2$ three-spin background. For this background, the corresponding gauge theory operators have been identified [9], and there exit some parameter regions where the fluctuations are stable [20, 29]. We develop a large J (total spin) expansion of the one-loop effective action, and compute it in a closed form up to and including $\mathcal{O}(1/J)$.

Collecting the contributions from the bosonic, the fermionic and the ghost sectors, the one-loop effective action is given by

$$e^{i\Gamma^{(1)}} = \frac{\text{Det}(\hat{\Delta}_F^+) \text{Det}(\hat{\Delta}_F^-) \text{Det}(-\partial^2)}{\text{Det}^{\frac{1}{2}}(D_{pq}^B) \text{Det}^{\frac{1}{2}}(\hat{D}_{mn}^B)}, \quad (4.1)$$

where Det stands for the functional determinant. To develop a large J expansion, we expand the fluctuation operators with respect to ν with ∂_i and m fixed:

$$\begin{aligned} \partial^2 + \kappa^2 &= (\partial^2 + \nu^2) + 2M^2, \\ \det \hat{D}_{mn}^B / \partial^2 &= (\partial^2 + \nu^2)^4 + 2\left[(2 - s_\gamma^2) + 2s_\gamma^2 \frac{\partial_\sigma^2}{\partial^2}\right] m^2 (\partial^2 + \nu^2)^3 \\ &\quad + 8c_\gamma^2 m^2 \partial_\sigma^2 (\partial^2 + \nu^2)^2 + \dots, \\ \det \hat{\Delta}_F^\pm &= (\partial^2 + \nu^2)^2 + 2M^2 \partial^2 + 2\beta_{1\pm}^2 (\partial_\tau^2 + \partial_\sigma^2) + 2\nu^2 (M^2 + \beta_{1\pm}^2) + \dots \end{aligned} \quad (4.2)$$

Note that $J = \Sigma_s J_s \sim \sqrt{\lambda} \nu$ and $M^2 \sim 2m^2 J_1 / J$ for large ν . This expansion is also regarded as that with respect to the power of $\partial^2 + \nu^2$, or the power of winding number m . The validity

²Precisely, the world-sheet momenta here are integer moded, whereas those in [29] are half-integer moded: We have started with an $su(2)$ rotated coordinates [20], and do not need a σ -dependent rotation.

of the expansion becomes clear shortly. Denoting the subleading terms for $\det \hat{D}_{mn}^B/\partial^2$ and $\det \hat{\Delta}_F^\pm$ by $\delta \hat{D}_{mn}^B$ and $\delta \hat{\Delta}_F^\pm$, respectively, and plugging (4.2) into (4.1), we find that the leading terms cancel each other. This is a consequence of the asymptotic BPS condition [21], or confirms the result for the point-like BMN solution. We are then left with

$$i\Gamma^{(1)} = \sum_{\eta=\pm} \text{Tr} \log \left(1 + \frac{\delta \hat{\Delta}_F^\eta}{[\partial^2 + \nu^2]^2} \right) - 2 \text{Tr} \log \left(1 + \frac{2M^2}{\partial^2 + \nu^2} \right) - \frac{1}{2} \text{Tr} \log \left(1 + \frac{\delta \hat{D}_{mn}^B}{[\partial^2 + \nu^2]^4} \right). \quad (4.3)$$

We would like to evaluate these terms by expanding the logarithms around unity. To check if that is possible, we first consider functional traces of the form

$$\text{Tr} \frac{(\partial_\tau^2)^a (\partial_\sigma^2)^b}{(\partial^2)^c (\partial^2 + \nu^2)^d}. \quad (4.4)$$

Since we are working with a Lorentzian world-sheet, we define the trace by the standard $i\epsilon$ prescription. We denote the volumes of the world-sheet time and space by T and $2\pi L$, the derivatives ∂_τ and ∂_σ in the momentum space by $i\omega$ and $ip_n = in/L$ ($n \in \mathbf{Z}$), respectively. Then, the trace reads $iT \sum_n \int \frac{d\omega}{2\pi i}$, and the ω -integral here picks up the poles on the negative real axis. If one can approximate the summation of p_n by an integral, one easily finds the leading large ν behavior of the trace in (4.4) to be

$$TL \frac{1}{\nu^{2(c+d-a-b-1)}}. \quad (4.5)$$

Here, we have to be a little careful about the infrared behavior: The approximation by an integral requires that the derivatives of the resultant integrand do not grow too large, but this may not hold for $p_n \sim 0$ because of the operator $1/\partial^2$. Fortunately, in our case, this “massless” operator appears always in the combination $\partial_\sigma^2/\partial^2$. From this, one can confirm, by evaluating the most singular terms for $p_n \sim 0$, that the infrared behavior does not spoil the scaling in (4.5) for the terms which we consider. Using this scaling, we find that the expansion of the logarithms in (4.3) around unity actually gives a large ν (or J) expansion.

More precisely, to evaluate the infrared behavior, we first divide the summation over n into two parts, i.e., that over $|n| \lesssim \nu L$ and $|n| \gtrsim \nu L$. The latter can be safely approximated by an integral by scaling ω, p_n to $\omega/\nu, p_n/\nu$. The contribution which can be most singular for $p_n \sim 0$ comes from the residue of the pole at $\omega = -p_n$, and behaves as $\sum_n \nu^{-2d} p_n^{1+2(a+b-c)}$. Thus, for $2(c-a-b) - 1 \leq 0$, this is infrared finite and the approximation by an integral is allowed.

The validity of the expansion is confirmed also as follows: The $i\epsilon$ prescription implies that the functional trace is given by a continuation from the Euclidean case. Then, one can easily check that, for large ν , the “subleading” terms in (4.2) are indeed smaller than the leading terms (some powers of $(\partial^2 + \nu^2)$) irrespectively of the values of $\partial_{\tau,\sigma}^2$.

We remark that if we use D_{mn}^B and Δ_F^- instead of \hat{D}_{mn}^B and $\hat{\Delta}_F^-$, we encounter the operator $(\partial^2)^2 + 4\nu^2 \partial_\tau^2$ instead of $(\partial^2 + \nu^2)^2$ for the leading terms, and its infrared behavior prevent the expansion of the logarithms for the large J expansion. This is essentially the same problem as in [29] (see [30]). In other words, for small ∂_τ , $(\partial^2)^2 + 4\nu^2 \partial_\tau^2$ can be smaller than other “subleading” terms, which invalidates the expansion.

Also, in general, one has to be careful about conjugation of a differential operator D by a time-dependent operator in evaluating the determinant of D . However, the time-dependent operators in our case are harmless, since they are rotation operators (e.g., (3.11)) which are essentially the same as operators such as $e^{ic\tau}$ with c a constant. To see this, first note that such operators just shift a pair of characteristic frequencies by a constant: $\omega_n \rightarrow \omega_n \pm c$, and these shifts are irrelevant to $\log \text{Det } D = \text{Tr} \log D \sim \sum \omega_n$; the plus and minus shifts cancel each other. One can confirm this by a simple example. For example, by the conjugation

$$\text{using } P(\tau\nu), P^{-1}(\tau\nu) \begin{pmatrix} \partial^2 & 2\nu\partial_\tau \\ -2\nu\partial_\tau & \partial^2 \end{pmatrix} P(\tau\nu) = \text{diag}(\partial^2 + \nu^2, \partial^2 + \nu^2) \text{ and hence}$$

$$\text{Det}(\partial^4 + 4\nu^2 \partial_\tau^2) = \text{Det}(\partial^2 + \nu^2)^2. \quad (4.6)$$

The characteristic frequencies of $(\partial^4 + 4\nu^2 \partial_\tau^2)$ are $\omega_n = \sqrt{p_n^2 + \nu^2} \pm \nu$, whereas those of $(\partial^2 + \nu^2)^2$ are $\omega_n = \sqrt{p_n^2 + \nu^2}$ with double degeneracy. Thus, the sum of the characteristic frequencies is actually the same. Second, note that the rotation operators may not change the norm of the functions and hence the Hilbert space. Finally, in the actual calculation, we first put a cut-off for large $|\tau|$ following the standard procedure of field theory. Then, the volume of the time T is factored out as in (4.8). Therefore, subtleties about $\tau \rightarrow \pm\infty$, if any, may not be relevant to the final results.

We are now ready to evaluate $\Gamma^{(1)}$. Once the expansion of the logarithms is allowed, the problem reduces to an ordinary calculation of an effective action in field theory. In fact, our expansion is essentially the same as the large (mass)² $\sim \nu^2$ expansion which guarantees the decoupling of heavy particles. This enables us to systematically compute $\Gamma^{(1)}$ up to higher order terms in the $1/\nu$ expansion. One then finds that, up to and including $\mathcal{O}(1/\nu)$, the first expansion of the logarithms is enough, and the terms in the ellipses in (4.2) do not contribute. Denoting each term in (4.3) by I_F^\pm, I_{AdS_5} and I_{S^5} , respectively, they are

$$\begin{aligned} I_F^\pm &\sim iT \sum_{n \in \mathbf{Z}} \left[\frac{M^2}{\sqrt{p_n^2 + \nu^2}} + \frac{\nu^2 \beta_{1\pm}^2}{(p_n^2 + \nu^2)^{\frac{3}{2}}} \right], \quad I_{AdS_5} \sim iT \sum_{n \in \mathbf{Z}} \frac{-2M^2}{\sqrt{p_n^2 + \nu^2}}, \\ I_{S^5} &\sim iT \sum_{n \in \mathbf{Z}} \left[\frac{(s_\gamma^2 - 2)m^2/2}{\sqrt{p_n^2 + \nu^2}} + c_\gamma^2 m^2 \frac{p_n^2}{(p_n^2 + \nu^2)^{\frac{3}{2}}} - \frac{s_\gamma^2 m^2}{\nu^2} \left(\frac{p_n^2}{\sqrt{p_n^2 + \nu^2}} - |p_n| \right) \right], \end{aligned} \quad (4.7)$$

where $\beta_{1+}^2 \sim c_\gamma^2 m^2, \beta_{1-}^2 \sim 0$ at this order. Combining all, we arrive at

$$i\Gamma^{(1)} \sim -iTLM^2 \cdot C, \quad (4.8)$$

with

$$C = \frac{1}{L\nu^2} \sum_{n \in \mathbf{Z}} \left(\frac{p_n^2 + (\nu^2/2)}{\sqrt{p_n^2 + \nu^2}} - |p_n| \right). \quad (4.9)$$

One can check that C is ultraviolet finite, in accord with [41]. It is also possible to evaluate C in a closed form for large ν . Summing up the summand as indicated above (namely, first summing the terms in the parenthesis for given n and then summing over n) can be approximated by integration, and gives $C \sim 1/2$. We summarize the evaluation of C in the appendix.

Here, there may be some issues to be considered. One is about regularization: Although C is finite in the above combination, each term in (4.3) or (4.7) is divergent. (Recall that the final result is obtained by (infinite bosonic contributions) – (infinite fermionic contributions).) Thus, one needs to regularize them, and the value of C may change with a different regularization. For instance, if we adopt the zeta-function regularization for each divergent sum in (4.9), we obtain $C \sim -1/2$. (See the appendix.) Another related issue is about finite renormalization: Since there are infinities in the intermediate steps, one may also need to take into account, in general, possible finite renormalization or contributions from finite counter terms. In a renormalizable theory, observables do not depend on regularization and renormalization procedure. Ambiguities of finite parts are fixed by appropriate renormalization conditions so as to maintain the symmetry. We will return to these issues at the end of section 7.

5 $SU(2)$ sector

In this section, we consider a class of solutions which is obtained by setting in (3.1) and (3.2)

$$a_3 = w_3 = m_3 = 0 \quad (5.1)$$

(with $J_1 \neq J_2$ generally). This has two spins in S^5 , and is called the $SU(2)$ sector. The corresponding gauge theory operators have been identified in [14]. Since $w_3 = m_3 = 0$ but $\nu \neq 0$ generally, one may not use relations such as $w_3^2 = \nu^2 + m_3^2$, which have been used so far (even before specializing to the $J_1 = J_2$ three-spin case). Thus, we repeat the same procedure from the beginning for this much simpler case, and just display the results.

Let us begin with the bosonic sector. The fluctuations in the AdS_5 part are described by (3.5), and its determinant by (3.6). For the S^5 part, the quadratic operator takes the form

$$D_{mn}^B = \delta_{mn} \partial^2 + 2(\omega_{\tau, mn} \partial_\tau - \omega_{\sigma, mn} \partial_\sigma) + N_{mn}, \quad (5.2)$$

where the non-vanishing $\omega_{i, mn}$ are $\omega_{i, \phi_1 \psi} = -a_2 \partial_i \phi_1$, $\omega_{i, \phi_2 \psi} = a_1 \partial_i \phi_2$; N_{mn} is a diagonal matrix with $N_{\gamma\gamma} = N_{\phi_3 \phi_3} = \nu^2$ and others zero. This is different from the corresponding

operator in (3.7) by this mass term (in addition to the form of $\omega_{i,mn}$). Its determinant is

$$\det D_{mn}^B = \partial^2(\partial^2 + \nu^2)^2[\partial^4 + a_1^2\Omega_2^2 + a_2^2\Omega_1^2]. \quad (5.3)$$

Since D_{mn}^B is non-trivial only for $m, n = (\phi_1, \phi_2, \psi)$, we first focus on this part. Similarly to the previous case, we then make a transformation of the type (3.13) with

$$Q_n^m = \frac{1}{\tilde{w}} \begin{pmatrix} a_1 w_2 & a_2 w_1 & 0 \\ a_2 w_1 & -a_1 w_2 & 0 \\ 0 & 0 & \tilde{w} \end{pmatrix}, \quad R_n^m = \begin{pmatrix} 1 & 0 \\ 0 & P(\tau\tilde{w}) \end{pmatrix}, \quad (5.4)$$

and $\tilde{w}^2 \equiv a_1^2 w_2^2 + a_2^2 w_1^2 = \nu^2 + a_1^2 m_2^2 + a_2^2 m_1^2$. This gives the same form of the quadratic operator as in (3.14) with $M_{mn} = \text{diag}(0, \tilde{w}^2, \tilde{w}^2)$. Combining it with the trivial part from $m, n = (\gamma, \phi_3)$, we find that

$$\det \hat{D}_{mn}^B = \partial^2(\partial^2 + \nu^2)^2[(\partial^2 + \tilde{w}^2)^2 + A_1 \frac{\partial_\sigma^2}{\partial^2}(\partial^2 + \tilde{w}^2) + A_2 \partial_\sigma^2], \quad (5.5)$$

where

$$A_1 = \left[\frac{2(m_1 w_1 + m_2 w_2)}{\tilde{w}} \right]^2, \quad A_2 = \left[\frac{2a_1 a_2 (m_1 w_2 - m_2 w_1)}{\tilde{w}} \right]^2. \quad (5.6)$$

Let us move on to the fermionic part. The quadratic fluctuations are described by the operator (2.11). We then make an $SO(1, 9)$ transformation as before. In this case, the corresponding element of $SO(1, 9)$ turns out to be given by (3.16) with the two-spin condition (5.1). (Note that l_s becomes $(0, 0, 1)$.) Proceeding similarly, one also finds that the pfaffian of the transformed operator \hat{D}_F is given by the same formulas, (3.25) and (3.26), with

$$\begin{aligned} \beta_{1\pm} &= \beta_{2\pm} = 0, \\ \beta_{3\pm} &= \mp \frac{a_1 a_2}{2MW} \kappa(m_1 w_1 - m_2 w_2), \quad \beta_{4\pm} = \mp \frac{a_1 a_2}{2MW} \kappa(m_1^2 - m_2^2). \end{aligned} \quad (5.7)$$

(M, W are defined as in three-spin case.) To simplify the expression, we further make a rotation (3.28), so that $\beta_{3\pm}$ are removed. We then arrive at

$$\det \hat{\Delta}_F^\pm = (\partial^2 + \nu^2)^2 + 2M^2 \partial^2 + 2\beta_4^2 (\partial_\tau^2 + \partial_\sigma^2) + (2\nu^2 + \beta_4^2 + M^2)(\beta_4^2 + M^2), \quad (5.8)$$

where $\beta_4^2 \equiv \beta_{4\pm}^2$.

Given these results, one can compute the one-loop effective action for the two-spin case. Up to and including $\mathcal{O}(1/\nu)$, it is given by the same formula as (4.8) and (4.9). In this case, $M^2 = a_1^2 m_1^2 + a_2^2 m_2^2 \sim \alpha(1 - \alpha)(m_1 - m_2)^2$, where $\alpha \equiv J_1/(J_1 + J_2)$.

The fluctuations of the $SU(2)$ case are not stable. However, we did not see any signals of instability. We also note that the results for the two-spin case with $J_1 = J_2$ are obtained by setting $c_\gamma = 0$ ($J_3 = 0$) in the $J_1 = J_2$ three-spin results. These suggest that our computation gives a smooth continuation from the stable case, at least at this order.

6 $SL(2)$ sector

There is another interesting class of simple solutions which has one spin in AdS_5 and one in S^5 [10]. This is called the $SL(2)$ sector. The fluctuations around these solutions are stable for large spins. The corresponding gauge theory operators have been identified in [34].

6.1 Solution

The constant radii solution in the $SL(2)$ sector is given by

$$Z_0 = a_0 e^{i\kappa\tau}, \quad Z_5 = a_5 e^{i(u\tau + k\sigma)}, \quad Z_1 = a_1 e^{i(w\tau + m\sigma)}, \quad (6.1)$$

where $a_0, a_1, a_5, \kappa, u, w$ are constant with $a_0^2 - a_5^2 = 1$ and $a_1 = 1$; k, m are integers (when the period of σ is 2π). Other fields are vanishing. The equations of motion give a constraint

$$u^2 = \kappa^2 + k^2. \quad (6.2)$$

Introducing $(W_5, W_1) \equiv (a_5 u, w)$ and $(M_5, M_1) \equiv (a_5 k, m)$, the Virasoro constraints read

$$a_0^2 \kappa^2 = W^2 + M^2, \quad 0 = W \cdot M. \quad (6.3)$$

The conserved charges are the space-time energy, $E = \sqrt{\lambda} a_0^2 \kappa$, and the spins in AdS_5 and S^5 , $S = \sqrt{\lambda} a_5^2 u$ and $J = \sqrt{\lambda} w$, respectively.

In the following, we are interested in the large spin limit with k and $\alpha \equiv S/J$ fixed. Since the number of the independent parameters are three, we are left with only one free parameter. We take $\nu^2 \equiv w^2 - m^2$ as this free parameter, and consider the large ν limit. Some useful relations are $m = -\alpha k$ and, for large ν ,

$$\begin{aligned} \kappa &\sim \nu \left[1 + \frac{k^2}{\nu^2} \alpha(\alpha + 1) \right], & u &\sim \nu \left[1 + \frac{k^2}{2\nu^2} (2\alpha^2 + 2\alpha + 1) \right], \\ w &\sim \nu \left[1 + \frac{k^2}{2\nu^2} \alpha^2 \right], & a_5^2 &\sim \alpha \left[1 - \frac{k^2}{2\nu^2} (1 + \alpha)^2 \right], \\ M^2 &\sim \alpha(1 + \alpha) k^2 \left[1 - \frac{k^2}{2\nu^2} (1 + \alpha) \right]. \end{aligned} \quad (6.4)$$

Note also that $J \sim \sqrt{\lambda} \nu$.

6.2 Bosonic fluctuation

From (2.5), it is straightforward to read off the bosonic fluctuation operator: The S^5 part is given by

$$D_{mn}^B = \text{diag}(\partial^2, \partial^2 + \nu^2, \partial^2 + \nu^2, \partial^2 + \nu^2, \partial^2 + \nu^2), \quad (6.5)$$

with $m, n = (\phi_1, \phi_2, \phi_3, \psi, \gamma)$, whereas the AdS_5 part is

$$\begin{aligned} D_{pq}^B &= \begin{pmatrix} D_{pq}^1 & 0 \\ 0 & D_{pq}^2 \end{pmatrix}, \\ D_{pq}^1 &= -\eta_{pq} \partial^2 + 2\omega_{i,pq} \partial^i, \quad D_{pq}^2 = \text{diag}(\partial^2 + \kappa^2, \partial^2 + \kappa^2), \end{aligned} \quad (6.6)$$

with $p, q = (t, \phi_5, \rho, \theta, \phi_4)$. The non-vanishing $\omega_{i,pq}$ are $\omega_{i,t\rho} = -a_5\partial_i t$ and $\omega_{i,\phi_5\rho} = a_0\partial_i\phi_5$. Thus,

$$\begin{aligned}\det D_{mn}^B &= \partial^2(\partial^2 + \nu^2)^4, \\ \det D_{pq}^B &= -\partial^2(\partial^2 + \kappa^2)^2[\partial^4 - 4a_5^2\kappa^2\partial_\tau^2 + 4a_0^2(u\partial_\tau - k\partial_\sigma)^2].\end{aligned}\quad (6.7)$$

These give the same characteristic frequencies as in [31].

For the large ν (or J) expansion, we further make an $SO(1, 2)$ rotation of the type (3.13) for the (t, ϕ_5, ρ) part. In this case,

$$Q_q^p = \frac{1}{\tilde{u}} \begin{pmatrix} a_0 u & a_5 \kappa & 0 \\ a_5 \kappa & a_0 u & 0 \\ 0 & 0 & \tilde{u} \end{pmatrix}, \quad R_q^p = \begin{pmatrix} 1 & 0 \\ 0 & P(-\tilde{u}\tau) \end{pmatrix}, \quad (6.8)$$

with $\tilde{u}^2 = a_0^2 u^2 - a_5^2 \kappa^2$ and $P(x)$ given in (3.11). Then, the fluctuation operator becomes

$$\hat{D}_{pq}^1 = -\eta_{pq}\partial^2 + M_{pq} - 2\rho_{\sigma,pq}\partial_\sigma, \quad (6.9)$$

where $M_{pq} = \text{diag}(0, \tilde{u}^2, \tilde{u}^2)$ and $\rho_{\sigma,pq}$ are defined similarly to the previous cases. Combining this with the trivial part from $p, q = (\theta, \phi_4)$, we find that

$$\det \hat{D}_{pq}^B = -\partial^2(\partial^2 + \kappa^2)^2[(\partial^2 + \tilde{u}^2)^2 - B_1 \frac{\partial_\sigma^2}{\partial^2}(\partial^2 + \tilde{u}^2) + B_2 \partial_\sigma^2], \quad (6.10)$$

with

$$B_1 = \left(\frac{2a_0 a_5 \kappa k}{\tilde{u}}\right)^2, \quad B_2 = \left(\frac{2a_0^2 u k}{\tilde{u}}\right)^2. \quad (6.11)$$

6.3 Fermionic fluctuation

Let us move on to the fermionic part. To simplify the kinetic term, we make an $SO(1, 9)$ rotation with

$$Q_b^a = \begin{pmatrix} q_b^a & 0 \\ 0 & 1_{7 \times 7} \end{pmatrix}, \quad q_b^a = \frac{1}{M} \begin{pmatrix} a_0 \kappa & -W_s \\ 0 & M_s \\ -W & a_0 \kappa W_s / W \end{pmatrix}, \quad (6.12)$$

where $a, b = (t, \phi_5, \rho, \theta, \dots)$ and $s = (5, 1)$. After this rotation, one finds the following fluctuation operator,

$$\hat{D}_F = \bar{\sigma}^i \partial_i + a \bar{\sigma}^{345} + c \bar{\sigma}^{023} + d \bar{\sigma}^{123}, \quad (6.13)$$

with

$$a = a_0 \kappa \frac{M_5}{M}, \quad c = \frac{a_5 \kappa}{2MW} (a_0^2 k^2 + M^2), \quad d = -\frac{a_0^2 a_5 \kappa u k}{2MW}. \quad (6.14)$$

Here, we have labelled $(t, \phi_5, \rho, \theta, \rho_4)$ as $(0, 1, 3, 4, 5)$ and (ϕ_1, \dots) as $(2, 6, 7, 8, 9)$.

With the explicit forms of σ^a in (3.21), \hat{D}_F takes a simple form

$$\hat{D}_F = \begin{pmatrix} \Delta_F^+ & 0 \\ 0 & \Delta_F^- \end{pmatrix} \otimes 1, \quad (6.15)$$

where

$$\Delta_F^+ = \begin{pmatrix} \Delta_1^+ & 0 \\ 0 & \Delta_2^+ \end{pmatrix}, \quad \Delta_1^+ = \begin{pmatrix} \partial_- + i(c+d) & a \\ -a & \partial_+ - i(c-d) \end{pmatrix}, \quad (6.16)$$

$\Delta_2^+(a, c, d) = \Delta_1^+(a, -c, -d)$, $\Delta_F^-(a, c, d) = \Delta_F^+(-a, c, d)$, and $\partial_\pm = \partial_\tau \pm \partial_\sigma$. From these, one finds that

$$\begin{aligned} \text{pf } \hat{D}_F &= \det \Delta_F^+ \det \Delta_F^-, \\ \det \Delta_F^+ &= \det \Delta_F^- \\ &= \left[\partial^2 + 2i(d\partial_\tau + c\partial_\sigma) + (c^2 - d^2 + a^2) \right] \left[\partial^2 - 2i(d\partial_\tau + c\partial_\sigma) + (c^2 - d^2 + a^2) \right]. \end{aligned} \quad (6.17)$$

This gives the characteristic frequencies

$$\omega_n = \epsilon_1 d + \epsilon_2 \sqrt{(n + \epsilon_1 c)^2 + a^2}, \quad (6.18)$$

with $\epsilon_1, \epsilon_2 = \pm 1$. Although the parameters (a, c, d) might look different at first from those in [31], one can show that they are actually the same (up to signs) by using relations among parameters.

It turns out that we do not have to perform further rotations as in the previous cases. However, it is useful to note that the terms with $id\partial_\tau$ can be removed by a simple “rotation”:

$$e^{\pm id\tau} [\partial^2 \pm 2i(d\partial_\tau + c\partial_\sigma) + (c^2 - d^2 + a^2)] e^{\mp id\tau} = \partial^2 + (a^2 + c^2) \pm 2ic\partial_\sigma. \quad (6.19)$$

6.4 One-loop effective action

To compute the one-loop effective action, we collect all contributions, including the ghost part, and expand the operators for large ν , to find that

$$\begin{aligned} i\Gamma^{(1)} &= -\text{Tr} \log \left(1 + \frac{b_1}{\partial^2 + \nu^2} \right) + 4 \text{Tr} \log \left(1 + \frac{b_2 + 2ic\partial_\sigma}{\partial^2 + \nu^2} \right) \\ &\quad - \frac{1}{2} \text{Tr} \log \left[1 + \frac{2b_3}{\partial^2 + \nu^2} + \frac{b_3^2 + B_2 \partial_\sigma^2}{(\partial^2 + \nu^2)^2} - B_1 \frac{\partial_\sigma^2}{\partial^2} \left(\frac{1}{\partial^2 + \nu^2} + \frac{b_3}{(\partial^2 + \nu^2)^2} \right) \right], \end{aligned} \quad (6.20)$$

with

$$b_1 = \kappa^2 - \nu^2, \quad b_2 = a^2 + c^2 - \nu^2, \quad b_3 = \tilde{u}^2 - \nu^2. \quad (6.21)$$

Then, up to and including $\mathcal{O}(1/\nu)$, the one-loop effective action in this case becomes

$$i\Gamma^{(1)} \sim +iTL M^2 \cdot C, \quad (6.22)$$

where $M^2 \sim \alpha(1+\alpha)k^2$ and C is given by (4.9).

7 Correction to space-time energy

Summarizing, up to and including $\mathcal{O}(1/\nu) \sim \mathcal{O}(1/J)$, the one-loop effective actions are given by a universal form,

$$i\Gamma^{(1)} \sim \mp iTL M^2 \cdot C, \quad (7.1)$$

with the minus sign for the $J_1 = J_2$ three-spin and the two-spin $SU(2)$ cases, and the plus sign for the $SL(2)$ case. We call the former two cases the S^5 case in the following. The parameter ν was defined in (3.2) for the S^5 case, and above (6.4) for the $SL(2)$ case, which was related to the total $SO(6)$ spin by $J \sim \sqrt{\lambda}\nu$ for large ν . T and $2\pi L$ were the volumes of the world-sheet directions τ and σ , respectively, whereas C was given by (4.9). M was given in terms of the winding numbers by equations below (3.16) and (6.2). It also has a unified geometrical expression $M^2 = \frac{1}{2}\eta^{ij}G_{\mu\nu}\partial_i x^\mu \partial_j x^\nu = \sqrt{-\det h_{ij}}$ with h_{ij} the induced metric. In terms of spins, this was expressed as

$$\begin{aligned} M^2 &= \frac{1}{2}\eta^{ij}G_{\mu\nu}\partial_i x^\mu \partial_j x^\nu \\ &\sim \begin{cases} 2m^2 J_1/(2J_1 + J_3) & (J_1 = J_2 \text{ 3-spin case}) \\ \alpha(1-\alpha)(m_1 - m_2)^2, \quad \alpha = J_1/(J_1 + J_2) & (SU(2) \text{ case}) \\ \alpha(1+\alpha)k^2, \quad \alpha = S/J & (SL(2) \text{ case}) \end{cases}. \end{aligned} \quad (7.2)$$

Formally, the results in (7.1) for the $SU(2)$ and the $SL(2)$ cases are related by replacing the filling fractions as $\alpha_{SU(2)} \rightarrow -\alpha_{SL(2)}$ with an identification of $m_1 - m_2$ and k .

7.1 Comparison with the gauge theory side

From these one-loop effective actions, one can obtain the corrections to the space-time energy $E^{(1)}$. First, let us note that, in general, a one-loop effective action is interpreted as the one-loop correction to the (world-volume) energy with the expectation values of fields fixed (see, e.g., [42]). A simple way to confirm this is to express the one-loop effective action as a summation over characteristic frequencies (as in (7.12)). In our case, this means that $\Gamma^{(1)}$ is proportional to the one-loop correction to the world-sheet energy:

$$\Gamma^{(1)} = -TE_{2d}^{(1)}. \quad (7.3)$$

Second, the correction to the world-sheet energy is translated into that of the space-time energy [43] as

$$E^{(1)} = \frac{1}{\kappa} E_{2d}^{(1)} , \quad (7.4)$$

because of $t = \kappa\tau$. Therefore, up to and including $\mathcal{O}(1/J^2)$,

$$E^{(1)} \sim \pm C \frac{M^2}{\kappa} + \mathcal{O}(1/J^3) \sim \pm \sqrt{\lambda} C \frac{M^2}{J} , \quad (7.5)$$

with the plus sign for the S^5 case, and the minus sign for the $SL(2)$ case. Here and in the following, we set $L = 1$. Comparing these with the classical expressions,

$$E^{(0)} = \begin{cases} J + \lambda \frac{M^2/2}{J} + \mathcal{O}(\lambda^2/J^3) & (S^5 \text{ case}) \\ S + J + \lambda \frac{M^2/2}{J} + \mathcal{O}(\lambda^2/J^3) & (SL(2) \text{ case}) \end{cases} , \quad (7.6)$$

one finds that the M^2 -dependence of $E^{(1)}$ is the same as that of the first subleading term in the classical expression.

One can also compare (7.5) with the gauge theory results. On the gauge theory side, the corresponding quantity is the $1/J$ correction to the anomalous dimension at order λ [32]-[36],

$$\gamma^{(1)} = \pm \frac{\lambda}{2J^2} \left[M^2 + K_{SL(2),SU(2)}(M^2) \right] , \quad (7.7)$$

where $K_{SL(2),SU(2)}(M^2)$ are certain functions of M^2 and called the “anomaly” terms [32, 36]. Thus, $E^{(1)}$ has the same M^2 -dependence of the first (“zero-mode”) term of $\gamma^{(1)}$. Note that, if C (+ possible contributions from finite counter terms) $\sim 1/\nu$, the J - and the λ -dependences also match: $E^{(1)} \sim \lambda M^2/J^2$. In addition, the zero-mode ($n = 0$) part of C , i.e., $1/2\nu$, gives exactly the same contribution to $E^{(1)}$ as the zero-mode term of $\gamma^{(1)}$.

7.2 Relation to the results in the literature

In the literature, the one-loop correction to the space-time energy has been studied by summing up the characteristic frequencies. It was computed numerically for the $J_1 = J_2$ three-spin case and the $SU(2)$ case with two equal spins in [30], and for the $SL(2)$ case in [31]. The result of the $SL(2)$ case was matched with the $1/J$ correction to the anomalous dimension (7.7) including the anomaly terms [32]. An agreement was also found in the $SU(2)$ case with two equal spins, up to subtleties due to the instability. Thus, in these cases,

$$E_{\text{literature}}^{(1)} = \gamma^{(1)} . \quad (7.8)$$

This is to be compared with our result (7.5). First, the zero-mode parts of $E_{\text{literature}}^{(1)}$ are the same as those on the gauge theory side in (7.7) [32] and hence as ours. However, the full

expression of $E_{\text{literature}}^{(1)}$ and the numerical results in [30] do not have a simple polynomial dependence on the winding numbers or M^2 . As for the J -dependence, if we evaluate C by a simple summation explained below (4.9), $C \sim 1/2$ and the J -dependence does not match, either. (However, see the discussions below (4.9) and (7.7).) Thus, the results in the literature and ours appear to be different. In this subsection, we would like to discuss this point in more detail.

First, we note that a one-loop effective action may be in general expressed either as a functional determinant (or a trace as in (7.9)), or as a summation over characteristic frequencies (as in (7.12)). Thus, our method is equivalent to those in [30, 31] before the summation. The agreement of the zero-mode contributions supports this. The apparent difference of the results is then understood as due to the difference of the methods of evaluation of the same quantity and extraction of its large J behavior. We have used the effective action language, since this is a useful tool for our purpose. In terms of it, the point of our method may be summarized as a choice of convenient functional bases for evaluating the one-loop effective action and space-time energy.

In order to see how apparent parameter dependence looks different depending on the way of evaluation and expansion, it may be useful to consider a simplified example of a one-loop effective action,³

$$i\Gamma^{(1)} = \text{Tr} \log(\partial^2 + \nu^2) + \text{Tr} \log(\partial^2 + \nu^2 + 2k^2) - 2 \text{Tr} \log(\partial^2 + \nu^2 + k^2), \quad (7.9)$$

with ν large and k of order one. Following our computations in the previous sections, one can combine all terms, to find

$$i\Gamma^{(1)} = \text{Tr} \log\left(1 + \frac{2k^2}{\partial^2 + \nu^2}\right) - 2 \text{Tr} \log\left(1 + \frac{k^2}{\partial^2 + \nu^2}\right). \quad (7.10)$$

The expansion of the logarithms gives a well-defined large ν expansion. For example, the first non-trivial term is

$$i\Gamma^{(1)} \sim \text{Tr} \frac{-k^4}{(\partial^2 + \nu^2)^2} = -\frac{iT}{4} \sum_n \frac{k^4}{(p_n^2 + \nu^2)^{3/2}} \sim -iT L \frac{k^4}{4\nu^2} \cdot c, \quad (7.11)$$

where we have approximated the sum by an integral, and $c = \int dx (x^2 + 1)^{-3/2} = 2$.

On the other hand, by a standard procedure, $\Gamma^{(1)}$ is written as a sum over characteristic frequencies as

$$i\Gamma^{(1)} \sim iT \sum_n \left(\sqrt{p_n^2 + \nu^2} + \sqrt{p_n^2 + \nu^2 + 2k^2} - 2\sqrt{p_n^2 + \nu^2 + k^2} \right), \quad (7.12)$$

up to an additive constant. Following [31, 32], one may expand each term with respect to $1/\nu^2$, to find

$$i\Gamma^{(1)} \sim iT\nu \sum_n \sum_{j=1} \frac{c_j(p_n, k)}{\nu^{2j}}, \quad (7.13)$$

³For the purpose of this subsection, one may take an even simpler example such as $\text{Tr} \log(\partial^2 + \nu^2 + k^2) - \text{Tr} \log(\partial^2 + \nu^2)$, although this is logarithmically divergent.

with $c_j(p_n, k)$ $2j$ -th polynomials of p_n and k . The parameter dependence of (7.13) is in fact very different from that of (7.11).

Here, we would like to make a comment on the expansion in (7.13). Although the original series is convergent, p_n in the summation can be larger than ν , and the terms with larger j become potentially more divergent.⁴ This subtlety can be rephrased also as follows. First, let us define

$$f(s) \equiv \Gamma^{(1)}/(T\nu) = \sum_n g_n(s), \quad (7.14)$$

where $s = 1/\nu^2$ and $g_n(s) = \sqrt{1 + sp_n^2} + \sqrt{1 + s(p_n^2 + 2k^2)} - 2\sqrt{1 + s(p_n^2 + k^2)}$. The above expansion corresponds to the expansion of $f(s)$ around $s = 0$ ($J = \infty$),

$$\begin{aligned} f(s) &= f(0) + sf'(0) + \cdots \\ &\sim s \sum_n g'_n(0). \end{aligned} \quad (7.15)$$

Note that, to obtain the last expression, one needs to use $f'(0) = \sum_n g'_n(0)$. By a basic result of analysis, a (sufficient) condition for such termwise differentiability is the uniform convergence of $\sum_n g'_n(s)$ around $s = 0$, in addition to the pointwise convergence of $\sum_n g_n(s)$ there. However, for large n , $g_n(s) \sim \sqrt{s}k^4/(p_n)^3$ and hence $g'_n(s) \sim k^4/(\sqrt{s}p_n^3)$, which implies that $\sum_n g'_n(s)$ is not uniformly convergent around $s = 0$. Since larger and larger n becomes relevant as s approaches zero, it seems to be difficult to study this subtlety numerically.⁵

In addition, to consider the relation between expansion methods and parameter dependence further, it may also be useful to observe what would happen if we did not make the rotations of the functional bases which we had performed to make the large J expansion well-defined. For example, in the $SL(2)$ case, if we use D_{pq}^1 in (6.6) instead of \hat{D}_{pq}^1 in (6.9), we have

$$-\frac{1}{2} \text{Tr} \log \left(1 + \frac{b'_1 \partial_\tau^2 + b'_2 \partial_\sigma^2 + b'_3 \partial_\tau \partial_\sigma}{\partial^4 + 4\nu^2 \partial_\tau^2} \right) \quad (7.16)$$

⁴In this example, $c_1 = 0$, whereas, in the original case in [31, 32], the first term in the expansion gives the finite leading correction. Regarding the divergences for $c_{j \geq 2}$, see a footnote in [32].

⁵Although the characteristic frequencies take different and more complicated forms, the expansion coefficients of $\Gamma^{(1)}$ in [31] are divergent as in (7.13), except in the leading term. Thus, the expansion is not defined. We also remark that the order of limits, i.e., $\lim_{N \rightarrow \infty} \sum_n^N$ first and then $\nu (\sim J) \rightarrow \infty$, has been changed in the expansions in [31] and (7.13). In the numerical computation in [30], a summation of the form $S = \sum_n f(n/\kappa)$ is first split into two parts as $S = (\sum_{|n| < N+1} + \sum_{|n| > N+1}) f(n/\kappa) \equiv S_1 + S_2$. S_1 is then numerically evaluated, whereas S_2 is regarded as the error. To estimate S_2 , an approximation by an integral in [29] is used: $f(x_i) = \int_{x_i}^{x_{i+1}} dx g(x) + \mathcal{O}(1/\kappa^5)$, where $g(x)$ is a certain function, and $x_i = n_i/\kappa$. Although S_2 is estimated in [30] as $S_2 = \int_{N/\kappa}^\infty dx g(x) + \mathcal{O}(1/\kappa^5)$, the above approximation of $f(x_i)$ by an integral is just for a single term $f(x_i)$ and thus the total error for S_2 derived from it is $\mathcal{O}(L/\kappa^5)$, where L is the number of terms in S_2 . Since $L = \infty$ in this case, the estimation of the error should be refined. In fact, the quantum corrections in Fig.1-5 in [30] at $q = 0$ ($s_\gamma^2 = 0$ in our notation) do not vanish, although it should vanish since $q = 0$ corresponds to the point-like BPS (BMN) solution.

instead of the second line in (6.20). Here, $b'_{1,2}$ are of order one, b'_3 is of order ν , and we have used (4.6). Suppose that one can expand the logarithm. Then, evaluating the ν -dependence similarly to section 4, one finds that there are infinitely many terms up to and including $\mathcal{O}(1/\nu)$ due to the infrared behavior. Thus, this expansion may not be valid in general. A formal expansion gives

$$i\Gamma^{(1)} \sim iT \frac{k^2}{\nu} [c'_1 + c'_2(kL)^2 + c'_3(kL)^4 \dots], \quad (7.17)$$

with c'_j some constants. The $E^{(1)}$ which is read off from this $\Gamma^{(1)}$ scales as $1/J^2$ and has complicated k^2 -dependence as in (7.7) with the anomaly terms.

We have discussed how the difference between our results and those in the earlier works arises. Actually, our work is an attempt to overcome the difficulties in the earlier works discussed above. In our method, the one-loop effective action and space-time energy can be computed as a systematic and well-defined expansion of $1/J$ including higher order terms. It is also possible to obtain closed forms of the expansion coefficients such as C in (4.9).

As discussed at the end of section 4, one has to regularize infinities in the intermediate calculations, and the final finite part should be fixed so as to maintain the symmetry of the theory. (Note that this issue is common to the methods in the earlier works.) In fact, one can confirm that our results maintain the symmetry: the final result is compactly expressed as a geometrical invariant (see (7.2)) and correctly vanishes in the BPS cases ($m, m_{1,2}, k = 0$). It is an interesting question if there are further possibilities to shift the finite part while keeping the symmetry maintained. Also, a very non-trivial agreement of the string and the gauge theory results is found in [32] based on [31], in spite of the difficulties in [31] concerning the large world-sheet momenta $n \gg J$. Therefore, the excitation modes with large n would be important to understand the differences among our results, those in [30, 31] and those in the gauge theory. Note that such high excitation modes are not well-understood in the context of AdS/CFT correspondence. For example, it is not known to what these correspond on the gauge theory side even in the pp-wave case.

8 Conclusions

In this paper, we discussed quantum fluctuations of the constant radii rotating strings in $AdS_5 \times S^5$. Using a functional method, we developed a systematic method to compute the one-loop sigma-model effective actions in closed forms as expansions for large spins. A point was the change (rotation) of functional bases to make the expansion well-defined. As examples, we explicitly evaluated the leading terms (up to and including $\mathcal{O}(1/J)$) for the strings in the $SO(6)$ sector with two equal spins, the $SU(2)$ sector, and the $SL(2)$ sector. We would like to note that, in our method, it is straightforward to compute higher order terms (up to any given order, in principle). We moreover obtained the one-loop corrections to the space-time energy up to and including $\mathcal{O}(1/J^2)$. Comparing these with the finite

size corrections to the anomalous dimension on the gauge theory side, we found that the dependence on the winding numbers and the filling fractions agreed with that of the “non-anomalous”(zero-mode) part on the gauge theory side. Relation to the earlier results in the literature was also discussed.

An obvious future direction is to probe quantum effects for more general cases such as the folded and the circular solutions in [6], by generalizing the method in this paper. In fact, this was one of the original motivations of this work. Also, the comparison with the results on the gauge theory side or in [31, 32] indicates that it is important to understand the issue of the order of limits, as has been pointed out in the literature of the rotating string/spin chain correspondence.

Acknowledgments

We would like to thank N. Berkovits, D. Ghoshal, N. Hatano, M. Hatsuda, Y. Hikida, T. Mateos, K. Mohri, N. Ishibashi, Y. Okada, S.-J. Rey, K. Sakai, M. Shiroishi, B. Stefanski, Y. Susaki, Y. Takayama, Z. Tsuboi, A. Yamaguchi, and K. Yoshida for useful discussions and conversations. We would especially like to thank A. Tseytlin for useful comments and discussions on an earlier version of the manuscript in January, correcting errors in it, and informing us of part of the results in [32] prior to publication. H. F. and Y.S. are also grateful to California Institute of Technology and Seoul National University, respectively, for warm hospitality. The work of Y.S. was supported in part by Grant-in-Aid for Young Scientists (B) No. 16740124 and Overseas Advanced Educational Research Practice Support Program No.16-077 from the Japan Ministry of Education, Culture, Sports, Science and Technology.

Appendix : Evaluation of C

In this appendix, we explain how to evaluate C in (4.9) for large ν . As discussed in section 4, the result can depend on the way of calculation (regularization).

First, we evaluate C by first summing the terms in the parenthesis in (4.9) for given n and then summing over n . This sum can be approximated by an integral using the Euler-Maclaurin formula,

$$\begin{aligned} \sum_{n=0}^N f(a + n\delta) \\ = \frac{1}{\delta} \int_a^{a+N\delta} f(x) dx + \frac{1}{2}[f(a) + f(a + N\delta)] + \frac{\delta}{12}[f'(a + N\delta) - f'(a)] + \mathcal{O}(\delta^2 f''), \end{aligned} \tag{A.1}$$

where the prime stands for the derivative. Setting $\delta = 1/(\nu L)$ and $f(x) = \delta \cdot (\frac{x^2+1/2}{\sqrt{x^2+1}} - x)$, we then find that, for large N ,

$$C \sim -\frac{\delta}{2} + 2 \sum_{n=0}^N f(n\delta) = +\frac{1}{2} + \frac{1}{6(\nu L)^2} + \mathcal{O}\left(\frac{1}{(\nu L)^3}, \frac{\nu L}{N}\right). \quad (\text{A.2})$$

Since $f''(x)$ is bounded for $x \in (0, \infty)$, the error does not grow, as discussed in section 4.

Second, for example, we may also adopt the zeta-function regularization (see, e.g., [44]) for each divergent sum in C . For this purpose, we introduce

$$\zeta(z, \mu) \equiv \sum_{n \in \mathbf{Z}} \frac{1}{(n^2 + \mu^2)^z} = \frac{1}{\Gamma(z)} \int_0^\infty ds s^{z-1} e^{-\mu^2 s} \theta_3(is/\pi), \quad (\text{A.3})$$

where $\mu > 0$ and $\theta_3(\tau) = \sum_{n \in \mathbf{Z}} q^{n^2/2}$ with $q = e^{2\pi i \tau}$. The second equation gives an analytic continuation in z . We then make a change of variables, $u = \mu^2 s$, perform the modular transformation $\theta_3(\tau) = (-i\tau)^{-1/2} \theta_3(-1/\tau)$, and expand the resultant theta function as $\theta_3(i\pi\mu^2/u) = 1 + 2 \sum_{n=1}^\infty e^{-(\pi\mu n)^2/u}$. After the integration, the terms from the sum over $n \geq 1$ are expressed by a Bessel function as $\frac{2\sqrt{\pi}}{\Gamma(z)} (\mu/\pi n)^{1/2-z} K_{1/2-z}(2\pi\mu n)$. These are exponentially suppressed for large μ , since $K_\nu(x) \sim x^{-1/2} e^{-x}$ for large x . Thus, we find that

$$\zeta(z, \mu) = \mu^{1-2z} B\left(\frac{1}{2}, z - \frac{1}{2}\right) + \mathcal{O}(\mu^{-z} e^{-2\pi\mu}) \quad (\text{A.4})$$

for large μ , where $B(x, y)$ is the beta function. Applying this to each sum in C , we encounter some singular expressions at intermediate steps. We regularize them by shifting z as $z + \epsilon$. Consequently,

$$C \sim \frac{1}{(\nu L)^2} \left[\zeta\left(-\frac{1}{2} + \epsilon, \nu L\right) - \frac{1}{2} (\nu L)^2 \zeta\left(\frac{1}{2} + \epsilon, \nu L\right) - 2\zeta(-1) \right]. \quad (\text{A.5})$$

Carefully taking the limit $\epsilon \rightarrow 0$, we find that the leading contributions from the first two terms cancel each other, and

$$C \sim -\frac{1}{2} + \frac{1}{6(\nu L)^2} + \mathcal{O}\left(\frac{e^{-2\pi\nu L}}{\sqrt{\nu L}}, \epsilon\right). \quad (\text{A.6})$$

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